GENERALIZED NEWMAN PHENOMENA AND DIGIT CONJECTURES ON PRIMES

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Abstract. We prove that the ratio of the Newman sum over numbers multiple of a fixed integer which is not multiple of 3 and the Newman sum over numbers multiple of a fixed integer divisible by 3 is o(1) when the upper limit of summing tends to infinity. We also discuss a connection of our results with a digit conjecture on primes.

1. Introduction

Denote for $x, \quad m \in \mathbb{N}$

$$S_m(x) = \sum_{0 \le n < x, n \equiv 0 \pmod m} (-1)^{s(n)}, \tag{1}$$

where $s(n)$ is the number of 1's in the binary expansion of n. Sum (1) is a *Newman digit sum.*

From the fundamental paper of A.O.Gelfond [4] it follows that

$$S_m(x) = O(x^{\lambda}), \quad \lambda = \frac{\ln 3}{\ln 4}. \tag{2}$$

The case $m = 3$ was studied in detail [5], [2], [7].

So, from the Coquet's theorem [2], [1] it follows that

$$-\frac{1}{3} + \frac{2}{\sqrt{3}} x^{\lambda} \le S_3(3x) \le \frac{1}{3} + \frac{55}{3}\left(\frac{3}{65}\right)^{\lambda} x^{\lambda} \tag{3}$$

with a microscopic improvement [7]

$$\frac{2}{\sqrt{3}} x^{\lambda} \le S_3(3x) \le \frac{55}{3}\left(\frac{3}{65}\right)^{\lambda} x^{\lambda}, \quad x \ge 2, \tag{4}$$

and moreover,

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$$\left\lfloor 2\left(\frac{x}{6}\right)^{\lambda}\right\rfloor \le S_3(x) \le \left\lceil \frac{55}{3}\left(\frac{x}{65}\right)^{\lambda}\right\rceil . \tag{5}$$

These estimates give the most exact modern limits of the so-called *Newman phenomena*. Note that M.Drmota and M.Skalba [3] using a close function ($S_m^{(m)}(x)$) proved that if m is a multiple of 3 then for sufficiently large x,

$$S_m(x) > 0, \quad x \ge x_0(m). \tag{6}$$

In this paper we study a general case for $m \ge 5$ (in the cases of $m = 2$ and $m = 4$ we have $|S_m(n)| \le 1$).

To formulate our results put for $m \ge 5$

$$\lambda_m = 1 + \log_2 b_m, \tag{7}$$

$$\mu_m = \frac{2b_m + 1}{2b_m - 1}, \tag{8}$$

where

$$b_m^2 = \begin{cases} \sin(\frac{\pi}{3}(1+\frac{3}{m}))(\sqrt{3} - \sin(\frac{\pi}{3}(1+\frac{3}{m}))), & if \ \ m \equiv 0 \pmod 3 \\ \sin(\frac{\pi}{3}(1-\frac{1}{m}))(\sqrt{3} - \sin(\frac{\pi}{3}(1-\frac{1}{m}))), & if \ \ m \equiv 1 \pmod 3 \\ \sin(\frac{\pi}{3}(1+\frac{1}{m}))(\sqrt{3} - \sin(\frac{\pi}{3}(1+\frac{1}{m}))), & if \ \ m \equiv 2 \pmod 3. \end{cases} \tag{9}$$

Directly one can see that

$$\frac{\sqrt{3}}{2} > b_m \ge \begin{cases} 0.86184088\ldots, & if \ \ (m,3) = 1, \\ 0.85559967\ldots, & if \ \ (m,3) = 3, \end{cases} \tag{10}$$

and thus,

$$\lambda_m < \lambda \tag{11}$$

and

$$3.73205080\ldots < \mu_m \le \begin{cases} 3.76364572\ldots, & if \ \ (m,3) = 1, \\ 3.81215109\ldots, & if \ \ (m,3) = 3. \end{cases} \tag{12}$$

Below we prove the following results.

Theorem 1. *If* $(m,3)=1$ *then*

$$|S_m(x)| \le \mu_m x^{\lambda_m}. \tag{13}$$

Theorem 2. *(Generalized Newman phenomena). If* $m>3$ *is a multiple of 3 then*

$$\left|S_m(x) - \frac{3}{m} S_3(x)\right| \le \mu_m x^{\lambda_m}. \tag{14}$$

Using Theorem 2 and (5) one can estimate $x_0(m)$ in (6). E.g., one can prove that $x_0(21) < e^{985}$.

2. Explicit formula for $S_m(N)$

We have

$$S_m(N) = \sum_{n=0, m|n}^{N-1} (-1)^{s(n)} = \frac{1}{m} \sum_{t=0}^{m-1} \sum_{n=0}^{N-1} (-1)^{s(n)} e^{2\pi i (\frac{nt}{m})} =$$

$$= \frac{1}{m} \sum_{t=0}^{m-1} \sum_{n=0}^{N-1} e^{2\pi i (\frac{t}{m} n + \frac{1}{2} s(n))}. \tag{15}$$

Note that the interior sum has the form

$$F_\alpha(N) = \sum_{n=0}^{N-1} e^{2\pi i (\alpha n + \frac{1}{2} s(n))} \quad 0 \le \alpha < 1. \tag{16}$$

Lemma 1. *If* $N = 2^{\nu_0} + 2^{\nu_1} + \ldots + 2^{\nu_r}$, $\nu_0 > \nu_1 > \ldots > \nu_r \ge 0$, *then*

$$F_\alpha(N) = \sum_{h=0}^{r} e^{2\pi i (\alpha \sum_{j=0}^{h-1} 2^{\nu_j} + \frac{h}{2})} \prod_{k=0}^{\nu_h - 1} (1 + e^{2\pi i (\alpha 2^k + \frac{1}{2})}), \tag{17}$$

where as usual $\sum_{j=0}^{-1} = 0$, $\prod_{k=0}^{-1} = 1$.

Proof. Let $r=0$. Then by (16)

$$F_\alpha(N) = \sum_{n=0}^{N-1} (-1)^{s(n)} e^{2\pi i \alpha n} = 1 - \sum_{j=0}^{\nu_0 - 1} e^{2\pi i \alpha 2^j} +$$

$$+ \sum_{0 \le j_1 < j_2 \le \nu_0 - 1} e^{2\pi i \alpha (2^{j_1} + 2^{j_2})} - \ldots = \prod_{k=0}^{\nu_0 - 1} (1 - e^{2\pi i \alpha 2^k}), \tag{18}$$

which corresponds to (17) for $r = 0$.

Assuming that (17) is valid for every N with $s(N) = r+1$ let us consider $N_1 = 2^{\nu_r}a + 2^{\nu_{r+1}}$ where a is odd, $s(a) = r+1$ and $\nu_{r+1} < \nu_r$. Let

$$N = 2^{\nu_r}a = 2^{\nu_0} + \ldots + 2^{\nu_r}; \quad N_1 = 2^{\nu_0} + \ldots + 2^{\nu_r} + 2^{\nu_{r+1}}.$$

Notice that for $n \in [0, 2^{\nu_{r+1}})$ we have

$$s(N+n) = s(N) + s(n).$$

Therefore,

$$F_\alpha(N_1) = F_\alpha(N) + \sum_{n=N}^{N_1-1} e^{2\pi i(\alpha n + \frac{1}{2}s(n))} =$$

$$= F_\alpha(N) + \sum_{n=0}^{2^{\nu_{r+1}}-1} e^{2\pi i(\alpha n + \alpha N + \frac{1}{2}(s(N)+s(n)))} =$$

$$= F_\alpha(N) + e^{2\pi i(\alpha N + \frac{1}{2}s(N))} \sum_{n=0}^{2^{\nu_{r+1}}-1} e^{2\pi i(\alpha n + \frac{1}{2}s(n))}.$$

Thus, by (17) and (18),

$$F_\alpha(N_1) = \sum_{h=0}^{r} e^{2\pi i(\alpha \sum_{j=0}^{h-1} 2^{\nu_j} + \frac{h}{2})} \prod_{k=0}^{\nu_h - 1} (1 + e^{2\pi i(\alpha 2^k + \frac{1}{2})} +$$

$$+ e^{2\pi i(\alpha \sum_{j=0}^{r} 2^{\nu_j} + \frac{r+1}{2})} \prod_{k=0}^{\nu_{r+1}-1} \left(1 + e^{2\pi i(\alpha 2^k + \frac{1}{2})}\right) =$$

$$= \sum_{h=0}^{r+1} e^{2\pi i(\alpha \sum_{j=0}^{h-1} 2^{\nu_j} + \frac{h}{2})} \prod_{k-0}^{\nu_h - 1} \left(1 + e^{2\pi i(\alpha 2^k + \frac{1}{2})}\right). \blacksquare$$

Formulas (15)-(17) give an explicit expression for $S_m(N)$ as a linear combination of the products of the form

$$\prod_{k=0}^{\nu_h-1} \left(1 + e^{2\pi i(\alpha 2^k + \frac{1}{2})}\right), \quad \alpha = \frac{t}{m}, \quad 0 \le t \le m-1. \tag{19}$$

Remark 1. *On can extract (17) from a very complicated general Gelfond formula [4], however, we prefer to give an independent proof.*

3. Proof of Theorem 1

Note that in (17)

$$r \leq \nu_0 = \left\lfloor \frac{\ln N}{\ln 2} \right\rfloor . \tag{20}$$

By Lemma 1 we have

$$|F_\alpha(N)| \leq \sum_{\nu_h=\nu_0,\nu_1,\ldots,\nu_r} \left| \prod_{k=1}^{\nu_h} \left(1 + e^{2\pi i(\alpha 2^{k-1}+\frac{1}{2})}\right) \right| \leq$$

$$\leq \sum_{h=0}^{\nu_0} \left| \prod_{k=1}^{h} \left(1 + e^{2\pi i(\alpha 2^{k-1}+\frac{1}{2})}\right) \right| . \tag{21}$$

Furthermore,

$$1 + e^{2\pi i(2^{k-1}\alpha+\frac{1}{2})} = 2\sin(2^{k-1}\alpha\pi)(\sin(2^{k-1}\alpha\pi) - i\cos(2^{k-1}\alpha\pi))$$

and, therefore,

$$\left| 1 + e^{2\pi i(2^{k-1}\alpha+\frac{1}{2})} \right| \leq 2 \left| \sin(2^{k-1}\alpha\pi) \right| . \tag{22}$$

According to (21) let us estimate the product

$$\prod_{k=1}^{h} (2|\sin(2^{k-1}\alpha\pi)|) = 2^h \prod_{k=1}^{h} |\sin(2^{k-1}\alpha\pi)|, \tag{23}$$

where by (15)

$$\alpha = \frac{t}{m}, \quad 0 \leq t \leq m-1. \tag{24}$$

Repeating arguments of [4], put

$$\left|\sin(2^{k-1}\alpha\pi)\right| = t_k. \tag{25}$$

Considering the function

$$\rho(x) = 2x\sqrt{1-x^2}, \quad 0 \leq x \leq 1, \tag{26}$$

we have

$$t_k = 2t_{k-1}\sqrt{1-t_{k-1}^2} = \rho(t_{k-1}). \tag{27}$$

Note that

$$\rho'(x) = 2(\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}) \leq -1 \tag{28}$$

for $x_0 \leq x \leq 1$, where

$$x_0 = \frac{\sqrt{3}}{2} \tag{29}$$

is the only positive root of the equation $\rho(x) = x$.
Show that either

$$t_k \leq \sin\left(\frac{\pi}{m}\left\lfloor\frac{m}{3}\right\rfloor\right) = \sin\left(\frac{\pi}{m}\left\lceil\frac{2m}{3}\right\rceil\right) = g_m < \frac{\sqrt{3}}{2} \tag{30}$$

or simultaneously $t_k > g_m$ and

$$t_k t_{k+1} \leq \max_{0\leq l\leq m-1}\left(\left|\sin\frac{l\pi}{m}\right|\left(\sqrt{3}-\left|\sin\frac{l\pi}{m}\right|\right)\right) =$$

(31)
$$= \begin{cases} \left(\sin\left(\frac{\pi}{m}\left\lfloor\frac{m}{3}\right\rfloor\right)\right)\left(\sqrt{3}-\sin\left(\frac{\pi}{m}\left\lfloor\frac{m}{3}\right\rfloor\right)\right), & if \quad m\equiv 1 \pmod 3 \\ \left(\sin\left(\frac{\pi}{m}\left\lceil\frac{m}{3}\right\rceil\right)\right)\left(\sqrt{3}-\sin\left(\frac{\pi}{m}\left\lceil\frac{m}{3}\right\rceil\right)\right), & if \quad m\equiv 2 \pmod 3 \end{cases} = h_m < \frac{3}{4}.$$

Indeed, let for a fixed values of $t \in [0, m-1]$ and $k \in [1, n]$

$$t2^{k-1} \equiv l(\mod m), \;\; 0 \leq l \leq m-1. \tag{32}$$

Then

$$t_k = \left|\sin\frac{l\pi}{m}\right|. \tag{33}$$

Now distinguish two cases: 1) $t_k \leq \frac{\sqrt{3}}{2}$ 2)$t_k > \frac{\sqrt{3}}{2}$.
In case 1)

$$t_k = \frac{\sqrt{3}}{2} \leftrightarrows \frac{l\pi}{m} = \frac{r\pi}{3}, \;\; (r,3)=1$$

and since $0 \le l \le m-1$ then

$$m = \frac{3l}{r}, \quad r = 1, 2.$$

Because of the condition $(m,3) = 1$, we have $t_k < \frac{\sqrt{3}}{2}$.
Thus, in (33)

$$l \in \left[0, \left\lfloor \frac{m}{3} \right\rfloor\right] \cup \left[\left\lceil \frac{2m}{3} \right\rceil, \ m\right]$$

and (30) follows.
In case 2) let $t_k > \frac{\sqrt{3}}{2} = x_0$. For $\varepsilon > 0$ put

$$1 + \varepsilon = \frac{t_k}{x_0} = \frac{2}{\sqrt{3}} \left|\sin(\pi 2^{k-1}\alpha)\right|, \tag{34}$$

such that

$$1 - \varepsilon = 2 - \frac{2}{\sqrt{3}} \left|\sin(\pi 2^{k-1}\alpha)\right|$$

and

$$1 - \varepsilon^2 = \frac{4}{3} \left|\sin(\pi 2^{k-1}\alpha)\right| \left(\sqrt{3} - \left|\sin(\pi 2^{k-1}\alpha)\right|\right). \tag{35}$$

By (27) and (34) we have

$$t_{k+1} = \rho(t_k) = \rho((1+\varepsilon)x_0) = \rho(x_0) + \varepsilon x_0 \rho'(c),$$

where $c \in (x_0, (1+\varepsilon)x_0)$.
Thus, according to (28) and taking into account that $\rho(x_0) = x_0$, we find

$$t_{k+1} \le x_0(1+\varepsilon)$$

while by (34)

$$t_k = x_0(1+\varepsilon).$$

Now in view of (35) and (29)

$$t_k t_{k+1} \le \left|\sin \pi 2^{k-1}\alpha\right| \left(\sqrt{3} - \left|\sin(\pi 2^{k-1}\alpha)\right|\right)$$

and according to (32),(33) we obtain that

$$t_k t_{k+1} \le h_m,$$

where h_m is defined by (31).

Notice that from simple arguments and according to (9)

$$g_m \le \sqrt{h_m} = b_m.$$

Therefore,

$$\prod_{k=1}^{h} \left|\sin(\pi 2^{k-1}\alpha)\right| \le (b_m^{\lfloor \frac{h}{2} \rfloor})^2 \le b_m^{h-1}.$$

Now, by (21)- (22), for $\alpha = \frac{t}{m}$, $t = 0, 1, ..., m-1$, we have

$$\left|F_{\frac{t}{m}}(N)\right| \le \sum_{h=0}^{\nu_0} \left|\prod_{k=1}^{h}(1 + e^{2\pi i(\alpha 2^{k-1} + \frac{1}{2})})\right| \le \sum_{h=0}^{\nu_0} 2^h \prod_{k=1}^{h} |\sin(2^{k-1}\alpha\pi)| \le$$

$$1 + 2\sum_{h=1}^{\nu_0}(2b_m)^{h-1} \le 1 + 2\frac{(2b_m)^{\nu_0}}{2b_m - 1}.$$

Note that, according to (7) and (20)

$$(2b_m)^{\nu_0} = 2^{\lambda_m \nu_0} \le 2^{\lambda_m \log_2 N} = N^{\lambda_m}.$$

Thus,

$$|F_{\frac{t}{m}}(N)| \le 1 + \frac{2}{2b_m - 1}N^{\lambda_m} \le \frac{2}{2b_m - 1 - \gamma_m}N^{\lambda_m},$$

where γ_m is defined by the equality

$$\frac{1}{2b_m - 1 - \gamma_m} - \frac{1}{2b_m - 1} = \frac{1}{2}.$$

Hence, we find

$$\gamma_m = \frac{(2b_m - 1)^2}{2b_m + 1}$$

and, consequently, by (8),

$$|F_{\frac{t}{m}}(N)| \le \frac{2b_m + 1}{2b_m - 1}N^{\lambda_m} = \mu_m N^{\lambda_m}.$$

Thus, the theorem follows from (15).■

4. Proof of Theorem 2.

Select in (15) the summands which correspond to $t = 0,\ \frac{m}{3},\ \frac{2m}{3}$.

We have

$$mS_m(N) = \sum_{n=0}^{N-1}\left(e^{\pi i s(n)} + e^{2\pi i(\frac{n}{3} + \frac{1}{2}s(n))} + e^{2\pi i(\frac{2n}{3} + \frac{1}{2}s(n))}\right) +$$

$$+ \sum_{t=1, t\ne \frac{m}{3}, \frac{2m}{3}}^{m-1} \ \sum_{n=0}^{N-1} e^{2\pi i(\frac{t}{m}n + \frac{1}{2}s(n))}. \tag{36}$$

Since the chosen summands do not depend on m and for $m=3$ the latter sum is empty then we find

$$(37)\qquad mS_m(N)=3S_3(N)+\sum_{t=1,t\neq\frac{m}{3},\frac{2m}{3}}^{m-1}\ \sum_{n=0}^{N-1}e^{2\pi i(\frac{t}{m}n+\frac{1}{2}s(n))}.$$

Further, the last double sum is estimated by the same way as in Section 3 such that

$$(38)\qquad \left|S_m(N)-\frac{3}{m}S_3(N)\right|\leq \mu_m N^{\lambda_m}\blacksquare.$$

Remark 2. *Notice that from elementary arguments it follows that if* $m\geq 5$ *is a multiple of* 3 *then*

$$\left(\sin\frac{\pi}{m}\left\lfloor\frac{m-1}{3}\right\rfloor\right)\left(\sqrt{3}-\sin\frac{\pi}{m}\left\lfloor\frac{m-1}{3}\right\rfloor\right)\leq$$
$$\leq\left(\sin\frac{\pi}{m}\left\lceil\frac{m+1}{3}\right\rceil\right)\left(\sqrt{3}-\sin\frac{\pi}{m}\left\lceil\frac{m+1}{3}\right\rceil\right).$$

The latter expression is the value of b_m^2 *in this case (see (9)).*

Example. Let us find some x_0 such that $S_{21}(x)>0$ for $x\geq x_0$. Supposing that x is multiple of 3 and using (4) we obtain that

$$S_3(x)\geq\frac{2}{3^{\lambda+\frac{1}{2}}}x^{\lambda}.$$

Therefore, putting $m=21$ in Theorem 2, we have

$$S_{21}(x)\geq\frac{1}{7}S_3(x)-\mu_{21}x^{\lambda_{21}}\geq\frac{2}{7\cdot 3^{\lambda+\frac{1}{2}}}x^{\lambda}-\mu_{21}x^{\lambda_{21}}.$$

Now, calculating λ and λ_m by (2) and (8), we find a required x_0 :

$$x_0=(3.5\cdot 3^{\lambda+\frac{1}{2}}\mu_{21})^{\frac{1}{\lambda-\lambda_{21}}}=e^{984.839\ldots}.$$

Corollary. *For* m *which is not a multiple of 3, denote* $U_m(x)$ *the set of the positive integers not exceeding* x *which are multiples of* m *and not multiples of 3. Then*

$$\sum_{n\in U_m(x)}(-1)^{s(n)}=-\frac{1}{m}S_3(x)+O(x^{\lambda_m}).$$

In particular, for sufficiently large x we have

$$\sum_{n\in U_m(x)} (-1)^{s(n)} < 0.$$

Proof. Since

$$|U_m(x)| = S_m(x) - S_{3m}(x)$$

then the corollary immediately follows from Theorems 1, 2.

5. On Newman sum over primes

In [6] we put the following binary digit conjectures on primes.
Conjecture 1. *For all $n \in \mathbb{N}, \quad n \neq 5, 6$*

$$\sum_{p\leq n}(-1)^{s(p)} \leq 0,$$

where the summing is over all primes not exceeding n.
Moreover, by the observations, $\sum_{p\leq n}(-1)^{s(p)} < 0$ beginning with $n = 31$.
Conjecture 2.

$$\lim_{n\to\infty} \frac{\ln\left(-\sum_{p\leq n}(-1)^{s(p)}\right)}{\ln n} = \frac{\ln 3}{\ln 4}.$$

A heuristic proof of Conjecture 2 was given in [8]. For a prime p, denote $V_p(x)$ the set of positive integers not exceeding x for which p is the least prime divisor. Show that the correctness of Conjectures 1 (for $n \geq n_0$) follows from the following very plausible statement, especially in view of the above estimates.

Conjecture 3. *For sufficiently large n we have*

$$\left|\sum_{5\leq p\leq\sqrt{n}} \sum_{j\in V_p(n), j>p} (-1)^{s(j)}\right| < \sum_{j\in V_3(n)} (-1)^{s(j)} = S_3(n) - S_6(n). \tag{39}$$

Indeed, in the "worst case" (really is not satisfied) in which for all $n \geq p^2$

$$\sum_{j\in V_p(n), j>p} (-1)^{s(j)} < 0, \quad p \geq 5. \tag{40}$$

we have a decreasing but *positive* sequence of sums

$$\sum_{j\in V_3(n),j>3}(-1)^{s(j)},\quad \sum_{j\in V_3(n),j>3}(-1)^{s(j)}+\sum_{j\in V_5(n),j>5}(-1)^{s(j)},$$
$$\ldots,\ \sum_{j\in V_3(n),j>3}(-1)^{s(j)}+\sum_{5\le p<\sqrt{n}}\ \sum_{j\in V_p(n),j>p}(-1)^{s(j)}>0.$$

Hence, the "balance condition" for odd numbers [8]

$$\left|\sum_{j\le n, j\ is\ odd}(-1)^{s(j)}\right|\le 1 \tag{41}$$

must be ensured permanently by the excess of the odious primes. This explains Conjecture 1.

It is very interesting that for some primes p most likely indeed (40) is satisfied for all $n\ge p^2$. Such primes we call "resonance primes". Our numerous observations show that all resonance primes not exceeding 1000 are:

$$11, 19, 41, 67, 107, 173, 179, 181, 307, 313, 421, 431, 433, 587,$$
$$601, 631, 641, 647, 727, 787.$$

In conclusion, note that for $p\ge 3$ we have

$$\lim_{n\to\infty}\frac{|V_p(n)|}{n}=\frac{1}{p}\prod_{2\le q<p}\left(1-\frac{1}{q}\right) \tag{42}$$

such that

$$\lim_{n\to\infty}\left(\sum_{p\ge 3}\frac{|V_p(n)|}{n}\right)=\frac{1}{2}. \tag{43}$$

Thus, using Theorems 1, 2 in the form

$$S_m(n)=\begin{cases}o(S_3(n)), & (m,3)=1\\ \frac{3}{m}S_3(n)(1+o(1)), & 3|m\end{cases} \tag{44}$$

and inclusion-exclusion for $p\ge 5$, we find

$$\sum_{j\in V_p(n)}(-1)^{\sigma(j)}=-\frac{3}{3p}\prod_{2\le q<p,\ \ q\ne 3}(1-\frac{1}{q})S_3(n)(1+o(1))=$$
$$-\frac{3}{2p}\prod_{2\le q<p}(1-\frac{1}{q})S_3(n)(1+o(1)). \tag{45}$$

Now in view of (5) we obtain the following absolute result as an approximation of Conjectures 1, 2.

Theorem 3. *For every prime number $p \geq 5$ and sufficiently large $n \geq n_p$ we have*

$$\sum_{j \in V_p(n)} (-1)^{s(j)} < 0$$

and, moreover,

$$\lim_{n \to \infty} \frac{\ln(-\sum_{j \in V_p(n)} (-1)^{s(j)})}{\ln n} = \frac{\ln 3}{\ln 4}.$$

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